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ABSTRACT

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Key Words and Phrases: difference, stationary, Gaussian, m-dependence,
axis-crossings, correlation functions.

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THE ASYMPTOTIC VARIANCE OF HIGHER ORDER CROSSINGS

By Benjamin Kedem and George Reed

1. Introduction

The computation of the variance of the number of axis crossings by a stationary process entails technical difficulties in both the continuous and the discrete time cases. What makes this problem so intractable is that the variance is expressed via quantities which involve conditional intensities and fourth order orthant probabilities which are difficult to compute. This is so even in the Gaussian case. See Cramer and Leadbetter (1967), Sec. 10.7, Cox and Isham (1980) pp.33,87, Kedem (1980) p.68. On the other hand when the process is sufficiently oscillatory, it is possible to estimate the variance under suitable conditions by a simple expression. This is shown here by examining the variance of the higher order crossings.

The notion of higher order crossings has been introduced by Kedem and Slud (1981), (1982) as follows. Let $\{Z_t\}_{t=-\infty}^{\infty}$ be a stationary discrete-time process and define a sequence of binary processes by

$$(1) \qquad X_t^{(k)} \equiv \begin{cases} 1 & \text{if } \nabla^{k-1} Z_t \geq 0 \\ 0 & \text{otherwise} \end{cases}, \qquad k = 1, 2, \ldots,$$

where ∇ is the backward-difference operator, $\nabla Z_t = Z_t - Z_{t-1}$. Then

$$(2) \qquad D_{k,N} \equiv \sum_{t=1}^{N-1} I[X_{t+1}^{(k)} \neq X_t^{(k)}]$$

is the number of axis-crossings by $\{\nabla^{k-1}Z_t\}_{t=1}^N$, and is called

the (number of) <u>higher order crossings of order</u> k <u>associated with</u>

$\{Z_t\}$. The main result of higher order crossings is (Kedem and

Slud (1982)) The

<u>Higher Order Crossings Theorem (HOCT)</u>: If $\{Z_t\}$ is a strictly

stationary process with finite variance and $P(Z_t \equiv Z_0 \forall t) = 0$ and

if the spectral distribution function F satisfies the condition

$dF(\pi) > 0$ then

(i) $\{X_t^{(k)}\} \Rightarrow \begin{cases} ...01010101... & \text{with probability } 1/2 \\ ...10101010... & \text{with probability } 1/2 \end{cases}$

(ii) $\lim\limits_{k \to \infty} \lim\limits_{N \to \infty} N^{-1} D_{k,N} = 1$ with probability 1.

Thus we see that $D_{k,N}$ eventually increases with k for sufficiently

large N. This monotone property of higher order crossings can be

associated with the amount of information they carry as exemplified

in Kedem and Slud (1981), where a discrimination statistic made of

higher order crossings is introduced. However the HOCT does not

tell us the rate of increase in the $D_{k,N}$. The present paper

provides a refinement of the HOCT in that we establish under some

conditions the rate of <u>decrease</u> in $\text{Var}(D_{k,N})$ as $k \to \infty$ for

fixed N. The fact that this variance should decrease is apparent

from the HOCT since the probability of two consecutive 1's in

$\{X_t^{(k)}\}_{t=1}^N$ approaches zero. More precisely, define the intensity

associated with $\{X_t^{(k)}\}$

(3) $\lambda_j^{(k)} \equiv P(X_t^{(k)} = 1 | X_{t-j}^{(k)} = 1), \quad j = 1, 2, \ldots ,$

Then our main result states that

Theorem 1. Let $\{Z_t\}$ be an m-dependent stationary Gaussian sequence. Then for fixed N

(4)
$$\lim_{k \to \infty} \frac{Var(D_{k,N})}{(N-1)\lambda_1^{(k)}} = 1.$$

The limit (4) certainly holds for any finite moving average. An explanation of (4) is provided by the intuition that by the HOCT, $\{X_t^{(k)}\}$ "becomes Markovian" and $\lambda_1^{(k)} \to 0$ as $k \to \infty$. Note that if $\{X_t^{(k)}\}$ is truly Markovian then $Var(D_{k,N}) = (N-1)\lambda_1^{(k)}(1-\lambda_1^{(k)})$ which yields (4) when $\lambda_1^{(k)}$ approaches zero. See Kedem (1980) Chapter 3.

Since for moderate and large k, $\{\nabla^{k-1}Z_t\}$ becomes oscillatory, i.e., $D_{k,N}$ is large, it is apparent from the HOCT that (4) sheds a great deal of light on the behavior of the variance of $D_{1,N}$ (number of axis crossings) for oscillatory processes. It should be noted that (4) is not entirely intuitive since in general we have

$$\frac{Var(D_{k,N})}{(N-1)\lambda_1^{(k)}} = \frac{(N-1)Var(\frac{D_{k,n}}{N-1})}{\lambda_1^{(k)}} \leq (N-1)(1-\lambda_1^{(k)})$$

for all k, and for $N \geq 2$. Other than $N = 2$, this general bound has little resemblance to (4). On the other extreme, many simulations with low order autoregressive moving average processes, at least for small k, suggested that the limit should be close to zero. It is only as k increases that (4) holds. It seems to us therefore that for large k, in light of our earlier comment, the

assumption that $\{X_t^{(k)}\}$ is approximately Markovian is not entirely unrealistic as it immediately leads to (4). In this paper however this assumption is not made.

The proof of Theorem 1 is carried out by first proving a series of lemmas. In Section 2 we establish a two sided inequality while in Section 3 we construct certain continuous differentiable functions which establish the rates of convergence of the correlations of $\{\nabla^k Z_t\}$ as $k \to \infty$.

2. An Inequality

In order to prove and motivate Theorem 1 we need to establish a few basic facts concerning axis crossings by Gaussian sequences. From now on it will be convenient to suppress k in $\lambda_j^{(k)}$ and in $X_t^{(k)}$ and we shall simply write λ_j and X_t with the understanding that both quantities depend on k.

First observe that $D_{k,N}$ can be written as a sum

$$(5) \qquad D_{k,N} = d_2 + d_3 + \ldots + d_N$$

where

$$(6) \qquad d_t = X_t + X_{t-1} - 2X_t X_{t-1}$$

and is also a function of k since X_t is.

Lemma 1. Let $\{Z_t\}_{t=-\infty}^{\infty}$ be a stationary Gaussian process. Then uniformly in t

$$(7) \qquad \mathrm{Cov}(X_{t-j}, d_t) = 0, \quad j = 0, \pm 1, \ldots .$$

Proof. The Gaussian assumption implies that

$$\mathrm{Cov}(X_{t-j}, X_t + X_{t-1} - 2X_t X_{t-1}) = [\tfrac{1}{2}\lambda_j - \tfrac{1}{4}] + [\tfrac{1}{2}\lambda_{j-1} - \tfrac{1}{4}]$$
$$- 2[\tfrac{1}{4}\lambda_1 + \tfrac{1}{4}\lambda_j + \tfrac{1}{4}\lambda_{j-1} - \tfrac{1}{4} - \tfrac{1}{4}\lambda_1] = 0$$

by noting that $\lambda_j = \tfrac{1}{2} + \tfrac{1}{\pi}\mathrm{Sin}^{-1}\rho_j$ where ρ_j is the correlation function of $\{Z_t\}$. (See Kedem (1980), p.33.) The orthogonality relation (7) implies that X_{t-j} and d_t are _independent_ since both variables are _binary_. We can therefore rewrite Lemma 1 as

Lemma 1'. Let $\{Z_t\}_{t=-\infty}^{\infty}$ be a stationary Gaussian process. Then

(8) $\qquad\qquad Cov(g(X_{t-j}), d_t) = 0, \quad j = 0, \pm 1, \ldots$

for any function g.

Next we examine the relation between $X_{t-j}X_{t-j+1}$ and d_t. Unlike the orthogonality relation (7) the covariance between the last two quantities cannot be evaluated directly but may be approximated. Without loss of generality we will focus on $Cov(d_2, X_t X_{t-1})$.

Lemma 2. Let $\{Z_t\}_{t=-\infty}^{\infty}$ be a stationary Gaussian process. Then we have the two sided inequality

(9) $\quad \max(\frac{1}{4}\lambda_{t-1} - \frac{1}{4}\lambda_{t-3} - \frac{1}{2}\lambda_1 + \frac{1}{2}\lambda_1^2, -\frac{1}{4}\lambda_{t-1} + \frac{1}{4}\lambda_{t-3} - \frac{1}{2}\lambda_1 + \frac{1}{2}\lambda_1^2)$

$\qquad\qquad \leq Cov(d_2, X_t X_{t-1}) \leq \frac{1}{2}\lambda_1^2.$

Proof. First note

$Cov(d_2, X_t X_{t-1}) = Ed_2 X_t X_{t-1} - Ed_2 E X_t X_{t-1} \leq (EX_t X_{t-1})(1 - Ed_2) = \frac{1}{2}\lambda_1^2.$

For the left hand side we have four cases corresponding to the "removal" of either X_1 or X_2 or X_t or X_{t-1}. Thus for example if we "remove" X_1 we have

$-2EX_1 X_2 X_{t-1} X_t \geq -2EX_2 X_{t-1} X_t = -2[-\frac{1}{4} + \frac{1}{4}\lambda_{t-3} + \frac{1}{4}\lambda_{t-2} + \frac{1}{4}\lambda_1]$

from which follows that

$$\text{Cov}(d_2, X_t X_{t-1}) \geq [-\tfrac{1}{4} + \tfrac{1}{4} \lambda_{t-1} + \tfrac{1}{4} \lambda_{t-2}] + [-\tfrac{1}{4} + \tfrac{1}{4} \lambda_{t-2} + \tfrac{1}{4} \lambda_{t-3}]$$

$$+ [\tfrac{1}{2} - \tfrac{1}{2} \lambda_{t-3} - \tfrac{1}{2} \lambda_{t-2} - \tfrac{1}{2} \lambda_1 + \tfrac{1}{2} \lambda_1^2]$$

$$= \tfrac{1}{4} \lambda_{t-1} - \tfrac{1}{4} \lambda_{t-3} - \tfrac{1}{2} \lambda_1 + \tfrac{1}{2} \lambda_1^2$$

and similarly for the other three cases.

Now observe that in general

(10) $$\text{Var}(d_t) = \lambda_1 (1 - \lambda_1)$$

and therefore by Lemma 1, Theorem 1 is proved if we can show that $\text{Cov}(d_s, X_t X_{t-1}) = \text{Cov}(d_s, d_t) = o(\lambda_1)$ for arbitrary t, s. However this is not immediate from Lemma 2 and requires some careful considerations to be discussed rigorously in the following Section. It should be made clear though that (9) is instrumental in proving that $\text{Cov}(d_s, d_t) = o(\lambda_1)$. We can support this claim by resorting to an example heuristic in nature.

Example 1. Assume X_t is Markovian. Then $\lambda_j = \tfrac{1}{2} + \tfrac{1}{2}(2\lambda_1 - 1)^j$ and so

$$\tfrac{1}{4}(\lambda_{t-1} - \lambda_{t-3}) = \tfrac{1}{2} \lambda_1 (\lambda_1 - 1)(2\lambda_1 - 1)^{t-3}.$$

Then the left hand side of (9) becomes

$$\text{Cov}(d_2, X_t X_{t-1}) \geq \max(\tfrac{1}{2} \lambda_1 (\lambda_1 - 1)(2\lambda_1 - 1)^{t-3} - \tfrac{1}{2} \lambda_1 + \tfrac{1}{2} \lambda_1^2,$$

$$-\tfrac{1}{2} \lambda_1 (\lambda_1 - 1)(2\lambda_1 - 1)^{t-3} - \tfrac{1}{2} \lambda_1 + \tfrac{1}{2} \lambda_1^2) = o(\lambda_1)$$

since $-\frac{1}{2}\lambda_1$ is always eliminated from <u>one</u> of these last two expressions regardless of whether t is odd or even. Thus for a Markovian X_t (9) and Lemma 1 imply

$$(11) \qquad\qquad Cov(d_2, d_t) = o(\lambda_1)$$

Another indication that (11) is plausible is furnished by the next numerical example.

<u>Example 2</u>. Assume $\{Z_t\}$ is a Gaussian white noise. Then

$$(12) \qquad\qquad \lambda_1 = \frac{1}{2} + \frac{1}{\pi}\, Sin^{-1}(\frac{-k}{k+1})\ .$$

and

$$(13) \qquad \lambda_2 = \frac{1}{2} + \frac{1}{\pi}\, Sin^{-1}\left[-1+2(\frac{k}{k+1})^2 + 2\left(\frac{2k+1}{(k+1)^2(k+2)}\right)\right].$$

Incidently, here we see the direct dependence of λ_1, λ_2 on k. Now define λ_2^* by

$$(14) \qquad\qquad \lambda_2^* = \frac{1}{2} + \frac{1}{\pi}\, Sin^{-1}[-1+2(\frac{k}{k+1})^2]$$

We maintain that as $k \to \infty$ λ_2^* approximates λ_2 very closely. This can be best seen from the following calculations.

k	λ_2	λ_2^*	λ_1	$\lambda_2 - \lambda_2^*$
10^3	0.97155556	0.97154135	0.001422932	0.00001421
5×10^3	0.98726994	0.98726865	0.00636567	0.00000129
10^4	0.99099766	0.99099720	0.00450140	0.00000046
10^5	0.99715295	0.99715295	0.00142353	0
10^6	0.99909968	0.99909968	0.00045017	0

We see that as $k \to \infty$ the white noise case implies

$$(15) \qquad \qquad \lambda_2 = \lambda_2^* + o(\lambda_1).$$

This can also be verified directly from (12)-(13). It follows that as $k \to \infty$

$$
\begin{aligned}
(16) \qquad \lambda_2 &= \frac{1}{2} + \frac{1}{\pi} \operatorname{Sin}^{-1}(-1+2\rho_1^2) + o(\lambda_1) \\
&= \frac{1}{2} + \frac{1}{\pi} \operatorname{Sin}^{-1}(-\cos 2\pi(\lambda_1 - \frac{1}{2})) + o(\lambda_1) \\
&= \frac{1}{2} - \frac{1}{\pi}[\frac{\pi}{2} - 2\pi(\frac{1}{2} - \lambda_1)] + o(\lambda_1) \\
&= 1 - 2\lambda_1 + o(\lambda_1).
\end{aligned}
$$

But in general, for any Gaussian stationary sequence

$$(17) \qquad \operatorname{Cov}(d_t, d_{t+1}) = -\frac{1}{2} + \lambda_1 + \frac{1}{2}\lambda_2 - \lambda_1^2.$$

and so, in the white noise case

$$\operatorname{Cov}(d_t, d_{t+1}) = o(\lambda_1) - \lambda_1^2 = o(\lambda_1).$$

This example indicates that we may also have $\operatorname{Cov}(d_t, d_s) = o(\lambda_1)$, $s \neq t$, for any two terms in (5). However the proof of this assertion requires the establishment of some further facts and the use of (9).

3. Proof that $Cov(d_t, d_s) = o(\lambda_1)$

As stated in Section 2 it is sufficient to show
$Cov(d_2, X_t X_{t-1}) = o(\lambda_1)$ in order to complete the proof of the
theorem. From (9) it is clear that $Cov(d_2, X_t X_{t-1}) \leq o(\lambda_1)$ so
the problem reduces to showing $o(\lambda_1) \leq Cov(d_2, X_t, X_{t-1})$ using
the left hand side of (9).

It is more convenient at this point to express $\lambda_t^{(k)}$ in terms
of the correlations of the kth differenced process, $\rho_{t,k}$. From
Kedem (1980) we have $\lambda_t^{(k)} = \frac{1}{2} + \frac{1}{\pi} \sin^{-1} \rho_{t,k}$. Ignoring the λ_1^2
term in (9) and suppressing the k, it will be sufficient to show

(18)
$$\lim_{k \to \infty} (\frac{1}{4\pi} \sin^{-1} \rho_{t-1} - \frac{1}{4\pi} \sin^{-1} \rho_{t-3} - \frac{1}{4} - \frac{1}{2\pi} \sin^{-1} \rho_1) / (\frac{1}{2} + \frac{1}{\pi} \sin^{-1} \rho_1) = 0$$

or

(19)
$$\lim_{k \to \infty} (-\frac{1}{4\pi} \sin^{-1} \rho_{t-1} + \frac{1}{4\pi} \sin^{-1} \rho_{t-3} - \frac{1}{4} - \frac{1}{2\pi} \sin^{-1} \rho_1) / (\frac{1}{2} + \frac{1}{\pi} \sin^{-1} \rho_1) = 0.$$

The solution involves two ideas. First treating all functions
of interest as functions of $\rho_{1,k}$ since this is the basic variable
at hand as is apparent from the HOCT, and secondly, circumventing
the problem that both numerator and denominator in (17), (18) go
to zero by constructing an equivalent limit with continuous differ-
entiable functions and using l'Hôpital's rule.

The following lemmas will show how to construct the equivalent
limit by replacing $\rho_{t,k}$ with an appropriate function. First
define the following notation associated with the k'th and
(k+1)'st differences.

$$\rho_{\nabla,k} = \rho_{1,k+1}$$

$$\rho_{\nabla,k}^{(r)} = \rho_{r,k+1}$$

$$R_{\nabla,k} = \frac{\rho_{\nabla,k+1} - \rho_{\nabla,k}}{\rho_{1,k+1} - \rho_{1,k}}$$

$C^1 \equiv C^1[-1, \rho_{1,1}]$ = set of continuously differentiable functions on the interval $[-1, \rho_{1,1}]$

Again, as above, whenever it is not confusing the subscript k is suppressed.

<u>Lemma 3</u>. If $\lim\limits_{k \to \infty} R_{\nabla,k}$ exists then for each sequence of ordered pairs $\{(\rho_{1,k}, \rho_{r,k})\}_{k=1}^{\infty}$, r fixed integer ≥ 2, there exists a function $\rho_r(\rho_1) \in C^1$ such that

(i) $\rho_r(\rho_{1,k}) \equiv \rho_{r,k}$.

(ii) $\lim\limits_{\rho_1 \to -1^+} \dfrac{d\rho_r}{d\rho_1} \equiv a_r$ is independent of the function constructed.

(iii) a_r satisfies the difference equation:

$$a_{r+1} - 2a_r(1 - 2a_1) + a_{r-1} = (-1)^r 2 \quad \text{where}$$

$$a_1 \equiv \lim\limits_{\rho_1 \to -1^+} \frac{d\rho_\nabla}{d\rho_1}.$$

$$a_0 \equiv 0.$$

Note that ρ_1 is $\rho_{1,k}$ but at the same time it is used as the variable of differentiation. Also, a_1 is defined differently from $a_2, a_3, \ldots$.

Proof. We first construct $\rho_\nabla(\rho_1)$ from the pairs $\{(\rho_{1,k}, \rho_{\nabla,k})\}_{k=1}^\infty$ and then the other functions will be produced inductively.

By a direct computation we have $\lim_{k \to \infty} \rho_{1,k} = -1$ and it has also been shown that $\{\rho_{1,k}\}_{k=1}^\infty$ is a strictly decreasing sequence (Kedem (1982)) to -1. In what follows the two equivalent notions of $k \to \infty$ and $\rho_1 \to -1$ are used interchangebly without loss of generality.

Since $\{\rho_{1,k}\}_{k=1}^\infty$ is strictly decreasing, for any finite set of values $\{(\rho_{1,k}, \rho_{\nabla,k})\}_{k=1}^n$ we can construct a function

$$\eta_\nabla^{(n)}(\rho_1) \in C^1[\rho_{1,n}, \rho_{1,1}] \quad \text{such that}$$

$$\eta_\nabla^{(n)}(\rho_{1,k}) = \rho_{\nabla,k} \quad \text{and}$$

$$\frac{d\eta_\nabla^{(n)}}{d\rho_1}(\rho_{1,k}) = R_{\nabla,k}.$$

If we let $\eta_\nabla^{(n+1)}(\rho_1)$ be the extension of $\eta_\nabla^{(n)}(\rho_1)$ then we can define

$$\rho_\nabla(\rho_1) = \lim_{n \to \infty} \eta_\nabla^{(n)}(\rho_1) \quad \text{for } \rho_1 \in (-1, \rho_{1,1}]$$

$$\text{and} \quad \rho_\nabla(-1) = \lim_{k \to \infty} \rho_\nabla(\rho_{1,k}) = \lim_{\rho_1 \to -1^+} \rho_\nabla(\rho_1)$$

$$\text{and} \quad \frac{d\rho_\nabla}{d\rho_1}(-1) = \lim_{k \to \infty} R_{\nabla,k} = \lim_{\rho_1 \to -1^+} \frac{d\rho_\nabla}{d\rho_1}.$$

By the construction of $\eta_\nabla^{(n)}(\rho_1)$, $\rho_\nabla(\rho_1) \in C'(-1, \rho_{1,1}]$ and since $\rho_\nabla(-1)$ is well defined by the HOCT and $\frac{d\rho_\nabla}{d\rho_1}(-1)$ is well defined by our assumption $\lim R_{\nabla,k}$ exists, then $\rho_\nabla(\rho_1) \in C'$.

To see that $\lim\limits_{\rho_1 \to -1^+} \dfrac{d\rho_\nabla}{d\rho_1}$ is independent of the construction

of $\rho_\nabla(\rho_1)$, note that any C^1 function satisfying (i) must

satisfy $\lim\limits_{\rho \to -1^+} \dfrac{d\rho_\nabla}{d\rho_1} = \lim\limits_{k \to \infty} R_{\nabla,k}$. This is so simply since by con-

struction for every $\varepsilon\, \exists K \ni$ for $k > K$ $\left| R_{\nabla,k} - \dfrac{d\rho_\nabla}{d\rho_1}(\rho_{1,k}) \right| < \varepsilon$.

And so $\lim\limits_{k \to \infty} R_{\nabla,k}$ is independent of the construction of ρ_∇.

We can now define the functions $\rho_r(\rho_1)$. Since

$\rho_\nabla = (2\rho_1 - \rho_2 - 1)/2(1-\rho_1)$ we have

$$\rho_{2,k} = -2(1-\rho_{1,k})\rho_{\nabla,k} + 2\rho_{1,k} - 1.$$

So we can define

(20) $\qquad \rho_2(\rho_1) = -2(1-\rho_1)\rho_\nabla(\rho_1) + 2\rho_1 - 1.$

In general, assume the existence of $\rho_{r-1}(\rho_1), \rho_{r-2}(\rho_1) \in C^1$. Then

from the fact that $\rho_\nabla^{(r)} = (2\rho_r - \rho_{r+1} - \rho_{r-1})/2(1-\rho_1)$ we obtain

$$\rho_{r,k} = -2(1-\rho_{1,k})\rho_{\nabla,k}^{(r-1)} + 2\rho_{r-1,k} - \rho_{r-2,k}$$

and we can define

(21) $\qquad \rho_r(\rho_1) = -2(1-\rho_1)\rho_\nabla^{r-1}(\rho_1) + 2\rho_{r-1}(\rho_1) - \rho_{r-2}(\rho_1)$

where $\rho_\nabla^{(r-1)}(\rho_1) \in C^1$ can be defined since we can always construct

a function $\bar\rho_\nabla^{(r-1)}(\rho_\nabla)$ using the same sequence of pairs and the same

constructions that we used for $\rho_{r-1}(\rho_1)$, so let $\rho_\nabla^{(r-1)}(\rho_1) = \bar\rho_\nabla^{(r-1)}(\rho_\nabla(\rho_1))$ a composition of C^1 functions.

Hence for each r there exists $\rho_r(\rho_1)$ satisfying (i) and (ii). To generate the difference equation in (iii) consider

$$\rho_{r+1}(\rho_1) = -2(1-\rho_1)\rho_\nabla^{(r)}(\rho_1) + 2\rho_r(\rho_1) - \rho_{r-1}(\rho_1).$$

Differentiate

$$\frac{d\rho_{r+1}}{d\rho_1} = 2\rho_\nabla^{(r)}(\rho_1) - 2(1-\rho_1)\frac{d\rho_\nabla^{(r)}}{d\rho_1} + 2\frac{d\rho_r}{d\rho_1} - \frac{d\rho_{r-1}}{d\rho_1}$$

and take the limit as $\rho_1 \to -1^+$

(22) $$a_{r+1} = 2(-1)^r - 4(\lim_{\rho_1 \to -1^+} \frac{d\rho_\nabla^{(r)}}{d\rho_1}) + 2a_r - a_{r-1}.$$

We can write $\dfrac{d\rho_\nabla^{(r)}}{d\rho_1} = \dfrac{d\rho_\nabla^{(r)}}{d\rho_\nabla}\dfrac{d\rho_\nabla}{d\rho_1}$ but note that

$$\lim_{\rho_1 \to -1^+}\frac{d\rho_\nabla^{(r)}}{d\rho_\nabla} = \lim_{\rho_1 \to -1^+}\frac{d\rho_r}{d\rho_1} \quad \text{or} \quad \lim_{\rho_1 \to -1^+}\frac{d\rho_\nabla^{(r)}}{d\rho_1} = a_r a_1 \quad \text{and therefore}$$

(22) becomes

$$a_{r+1} = 2(-1)^r - 4a_r a_1 + 2a_r - a_{r-1}$$

which can be rewritten to give the difference equation in (iii). This completes the proof of Lemma 3.

Lemma 4. If $\lim_{k\to\infty} R_{\nabla,k} = 1$ then $a_r = (-1)^{r+1} r^2$.

Proof. $\lim_{k\to\infty} R_{\nabla,k} = 1$ means $a_1 = 1$ so that the difference equation becomes:

$$a_{r+1} + 2a_r + a_{r-1} = (-1)^r 2.$$

A specific solution is $a_r = (-1)^{r+1} r^2$. The general solution

to the homogeneous equation $a_{r-1} + 2a_r + a_{r-1} = 0$ is

$$\alpha(-1)^r + \beta(-1)^r r.$$

But using (20) $a_2 = -4$ and using (21)(or simply (22) with $r = 2$) with $r = 3$ $a_3 = 9$. These conditions result in $\alpha = \beta = 0$ and thus $a_r = (-1)^{r+1} r^2$.

Lemma 5. $\lim\limits_{k \to \infty} R_{\nabla,k} = 1$ if the process is m-dependent.

Proof. The result can be derived from the expression for $\rho_{1,k}$ given in Kedem and Slud (1981):

$$(23) \qquad \rho_{1,k} = \frac{-\binom{2k}{k-1} + \rho_1\left[\binom{2k}{k} + \binom{2k}{k-2}\right] - \ldots + (-1)^k \rho_{k+1}}{\binom{2k}{k} - 2\rho_1\binom{2k}{k-1} + \ldots + (-1)^k 2\rho_k}.$$

For white noise, it is a straightforward computation since $\rho_{1,k} = -\frac{k}{k+1}$. In general if $\rho_i = 0$ for $i > m$, let $k > m$ so that there are no new correlation terms in (23) as k increases. Then $\rho_{1,k+1} - \rho_{1,k}$ can always be written as the quotient of two finite polynomials in k: $P_1(k)/P_2(k)$. Since there are no new terms in the expression, as k increases, then $\rho_{\nabla,k+1} - \rho_{\nabla,k}$ will be the quotient of the same two polynomials only in $k + 1$: $P_1(k+1)/P_2(k+1)$. Hence $R_{\nabla,k} = \frac{P_1(k+1)}{P_1(k)} \cdot \frac{P_2(k)}{P_2(k+1)}$ and since these are finite polynomials in k, $\lim\limits_{k \to \infty} R_{\nabla,k} = 1$

With these results, it is now possible to evaluate (18) and (19). If t is even consider (18). Using Lemma 3 we can write

an equivalent limit by replacing $\rho_{t,k}$ with $\rho_t(\rho_1)$ and considering:

(24)

$$\lim_{\rho_1 \to -1^+}\left(\frac{1}{4\pi}\sin^{-1}\rho_{t-1}(\rho_1) - \frac{1}{4\pi}\sin^{-1}\rho_{t-3}(\rho_1) - \frac{1}{4} - \frac{1}{2\pi}\sin^{-1}\rho_1\right)/\left(\frac{1}{2} + \frac{1}{\pi}\sin^{-1}\rho_1\right).$$

Using l'Hôpital's rule:

(25)
$$\lim_{\rho_1 \to -1^+}\frac{\dfrac{1}{4}\dfrac{1}{\sqrt{1-\rho_{t-1}^2}} - \dfrac{d\rho_{t-1}}{d\rho_1} - \dfrac{1}{4}\dfrac{1}{\sqrt{1-\rho_{t-3}^2}}\dfrac{d\rho_{t-3}}{d\rho_1} - \dfrac{1}{2}\dfrac{1}{\sqrt{1-\rho_1^2}}}{\dfrac{1}{\sqrt{1-\rho_1^2}}}.$$

This limit can be evaluated if we assume m-dependence. Using

Lemmas 4 and 5 we note that $\displaystyle\lim_{\rho_1 \to -1^+}\frac{1-\rho_1^2}{1-\rho_t^2} = \lim_{\rho_1 \to -1^+}\frac{-2\rho_1}{-2\rho_t\dfrac{d\rho_t}{d\rho_1}} = \frac{1}{|a_t|}$.

Hence, $\displaystyle\lim_{\rho_1 \to -1^+}\frac{\sqrt{1-\rho_1^2}}{\sqrt{1-\rho_t^2}} = \frac{1}{\sqrt{|a_t|}}$. Thus (25) becomes:

(26)
$$\frac{1}{4}\frac{a_{t-1}}{\sqrt{|a_{t-1}|}} - \frac{1}{4}\frac{a_{t-3}}{\sqrt{|a_{t-3}|}} - \frac{1}{2}.$$

When t is even

$$a_{t-1} = +(t-1)^2, \quad a_{t-3} = +(t-3)^2$$

and (26) equals

$$\frac{1}{4}(t-1) - \frac{1}{4}(t-3) - \frac{1}{2} = 0.$$

When t is odd a similar computation shows (19) reduces to zero as well. So for all t $\text{Cov}(d_2, X_t X_{t-1}) = o(\lambda_1)$. This completes the proof of the theorem.

Since by Lemma 5 $\lambda_1^{(k)}/\lambda_1^{(k+1)} \to 1$, $k \to \infty$, we also have

<u>Corollary 1</u>. Under the same conditions as in Theorem 1

$$\lim_{k \to \infty} \frac{\text{Cov}(D_{k,N}, D_{k+1,N})}{(N-1)\lambda_1^{(k)}} \le 1.$$

<u>Proof</u>. Use Cauchy-Schwarz inequality.

As the m-dependence assumption was used only in showing that $R_{\nabla,k} \to 1$, $k \to \infty$, it is clear that the theorem can be extended to cover a more general case for which this requirement still holds provided ρ_j approaches zero fast enough. We conjecture that Theorem 1 holds true at least for stationary autoregressive moving average Gaussian sequences. The proof of this requires a more careful examination of (23).

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